Arc Length Formula	**Explanation**
$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$	Arc length of "nice functions" with respect to x.

1. Find the length of the following curves.

   (a) $\frac{3}{4}x - \frac{1}{4}$, $-1 \leq x \leq 3$.

Solution:

We simply calculate

$$\begin{aligned} L &= \int_a^b \sqrt{1+[f'(x)]^2}\,dx \\ &= \int_{-1}^{3} \sqrt{1+\left[\frac{3}{4}\right]^2}\,dx \\ &= \int_{-1}^{3} \sqrt{\frac{25}{16}}\,dx \\ &= \frac{5}{4}x\bigg|_{-1}^{3} \qquad = 5. \end{aligned}$$

(b) $12x = 4y^3 + 3y^{-1}, \;\; 1 \leq y \leq 3.$

Solution:
Let us first find

$$f(y) = \frac{1}{3}y^3 + \frac{1}{4}y^{-1}$$
$$f'(y) = y^2 - \frac{1}{4}y^{-2}.$$

Now,

$$\begin{aligned} L &= \int_1^3 \sqrt{1 + \left[y^2 - \frac{1}{4}y^{-2}\right]^2}\, dx \\ &= \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16}y^{-4}}\, dx \\ &= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4}y^{-2}\right)^2}\, dx \\ &= \int_1^3 y^2 + \frac{1}{4}y^{-2}\, dx \\ &= \frac{1}{3}y^3 - \frac{1}{4}y^{-1}\Big|_1^3 \qquad = \frac{53}{6}. \end{aligned}$$

2. Find b such that the length of the curve $y = 1 + 6x^{3/2}$, $0 \leq x \leq b$, is 6.

Solution:

This time, we know $L = 6$, and we want to find b, so we just calculate as usual:

$$f(x) = 1 + 6x^{3/2}$$
$$f'(x) = 9x^{1/2}$$

$$\begin{aligned} 6 &= \int_0^b \sqrt{1 + 81x}\, dx \\ &= \frac{1}{81} \cdot \frac{2}{3}(1 + 81x)^{3/2}\Big|_0^b && \text{(using } u = 1 + 81x\text{)} \\ &= \frac{2}{81 \cdot 3}\left[(1 + 81b)^{3/2} - 1\right]. \end{aligned}$$

Solving for b gives

$$b = \frac{\left(\frac{6 \cdot 81 \cdot 3}{2} + 1\right)^{2/3} - 1}{81}.$$

Surface Area Formula	**Explanation**
$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$	Surface area of "nice functions" rotated about the y-axis.

3. Find the surface areas obtained in the following situations.
   (a) rotating the curve $f(x) = \sqrt{1+e^x}$ for $0 \le x \le 1$ around the x-axis.

Solution:
As usual, we need

$$f(x) = \sqrt{1+e^x},$$
$$f'(x) = \frac{1}{2}(1+e^x)^{-1/2} \cdot e^x.$$

Now, we compute

$$\begin{aligned} S &= \int_0^1 2\pi\sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\,dx \\ &= 2\pi\int_0^1 \sqrt{1+e^x}\sqrt{\frac{e^{2x}+4e^x+4}{4(1+e^x)}}\,dx \\ &= 2\pi\int_0^1 \sqrt{1+e^x}\sqrt{\frac{(e^x+2)^2}{4(1+e^x)}}\,dx \\ &= 2\pi\int_0^1 \sqrt{1+e^x}\frac{e^x+2}{2\sqrt{1+e^x}}\,dx \\ &= \pi\int_0^1 e^x+2\,dx \\ &= \pi\left(e^x+2x\right)\Big|_0^1 \qquad = \pi(e+1). \end{aligned}$$

(b) rotating the curve $f(y) = \sqrt{a^2 - y^2}$ for $0 \leq y \leq a/2$ and a constant around the y-axis

Solution:
And again,

$$f(y) = \sqrt{a^2 - y^2},$$
$$f'(y) = \frac{1}{2}(a^2 - y^2)^{-1/2} \cdot -2y = -\frac{y}{\sqrt{a^2 - y^2}}.$$

$$\begin{aligned}
S &= \int_0^{a/2} 2\pi\sqrt{a^2 - y^2}\sqrt{1 + \frac{y^2}{a^2 - y^2}}\,dx \\
&= 2\pi \int_0^{a/2} \sqrt{a^2 - y^2}\sqrt{\frac{a^2}{a^2 - y^2}}\,dx \\
&= 2\pi \int_0^{a/2} \sqrt{a^2 - y^2}\frac{a}{\sqrt{a^2 - y^2}}\,dx \\
&= 2\pi \int_0^{a/2} a\,dx \\
&= 2\pi\ (ax)\Big|_0^{a/2} \\
&= 2\pi\left(\frac{a^2}{2} - 0\right) \qquad = \pi a^2.
\end{aligned}$$